

# Section 7.1 Rational Expressions and Their Simplification

**Definition of a rational expression:** A rational expression is the quotient of two polynomials.

**Rule prohibiting division by zero:** If a is any number, then  $a \div 0$  is undefined.

**Excluding numbers from rational expressions:** If a variable in a rational expression is replaced by a number that causes the denominator to be 0, that number must be excluded as a replacement for the variable. The rational expression is undefined at any value that produces a denominator of 0.

To find the excluded values for a rational expression (the values for which the expression is undefined), set the denominator(s) equal to zero, solve, and those solutions are the excluded values. These values are values for which the rational expression is undefined.

**Example 1:** For each of the given expressions, find the excluded values (values for which the expression is undefined).

a.  $\frac{2x}{x-3}$       Solution:  $x-3=0$   
 $x=3$

3 is an excluded value (a value at which the rational expression is undefined)

b.  $\frac{7+x}{2x-5}$

Solve:  $2x-5=0$   
 $2x-5+5=0+5$   
 $\frac{2x}{2}=\frac{5}{2}$   
 $x=\frac{5}{2}$

$\frac{5}{2}$  is an excluded value.

c.  $\frac{2x-1}{x^2-x-6}$

Solve:  $x^2-x-6=0$   
 $(x-3)(x+2)=0$   
 Either  $x-3=0$ , or  $x+2=0$   
 $x=3$                        $x=-2$

3 and -2 are both excluded values.

d.  $\frac{7x+7}{x^2-9}$

Solve:  $x^2-9=0$   
 $(x-3)(x+3)=0$   
 Either  $x-3=0$ , or  $x+3=0$   
 $x=3$ , or  $x=-3$

3 and -3 are both excluded values.

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer.

**Simplifying rational expressions:**

**Rule:** If P, Q, and R are polynomials and Q and R are not 0, then

$$\frac{PR}{QR} = \frac{P}{Q}$$

Process for simplifying:

1. Factor the numerator and denominator completely.
2. Divide both the numerator and denominator by the common factors.

Example 2: Simplify each of the given expressions:

a.  $\frac{4x+28}{20x} = \frac{4(x+7)}{4 \cdot 5 \cdot x} = \frac{x+7}{5x}$

b.  $\frac{10x-15}{6x-9} = \frac{5 \cdot (2x-3)}{3 \cdot (2x-3)}$   
 $= \frac{5}{3}$

c.  $\frac{x^3-x^2}{x-1} = \frac{x^2 \cdot (x-1)}{1 \cdot (x-1)}$   
 $= \frac{x^2}{1}$   
 $= x^2$

SIDE WORK

$$10x-15 = 5 \cdot 2x - 5 \cdot 3$$

$$= 5 \cdot (2x-3)$$

$$6x-9 = 3 \cdot 2x - 3 \cdot 3$$

$$= 3 \cdot (2x-3)$$

$$x^3-x^2 = x^2 \cdot x - x^2 \cdot 1$$

$$= x^2 \cdot (x-1)$$

$$x-1 = 1 \cdot x - 1 \cdot 1$$

$$= 1 \cdot (x-1)$$

**Relationship between opposite factors:** If two polynomials are identical except that the sign of each term in one polynomial is opposite of the sign of that term in the other polynomial, then one polynomial is the negative of the other.

**Simplifying rational expressions with opposite factors in the numerator and denominator:** The quotient of two polynomials that have opposite signs and are additive inverses is  $-1$ .

Example 3: Simplify each of the following. If the numerator and the denominator contain opposite factors, the quotient of those factors is  $-1$ .

a.  $\frac{x-3}{3-x}$  Solution:  $\frac{x-3}{3-x} = \frac{x-3}{-1(x-3)} = \frac{1}{-1} = -1$

b.  $\frac{2x-1}{1-2x} = \frac{2x-1}{-2x+1} = \frac{1 \cdot (2x-1)}{-1 \cdot (2x-1)} = \frac{-1}{1} = -1$

c.  $\frac{4x^2-25}{15-6x} = \frac{4x^2-25}{-6x+15} = \frac{(2x-5)(2x+5)}{-3 \cdot (2x-5)} = \frac{2x+5}{-3}$

d.  $\frac{x^2-5x+6}{8-2x-x^2} = \frac{x^2-5x+6}{-x^2-2x+8} = \frac{(x-2)(x-3)}{-1 \cdot (x-2)(x+4)} = -\frac{x-3}{x+4}$

SIDEWORK

$1-2x = -2x+1$ $= -1 \cdot (2x-1)$
$4x^2-25 = (2x-5)(2x+5)$
$15-6x = -6x+15$ $= (-3)(2x) + (-3)(-5)$ $= -3(2x-5)$
$x^2-5x+6 = (x-2)(x-3)$
$8-2x-x^2 = -x^2-2x+8$ $= -1 \cdot (x^2+2x-8)$ $= -1 \cdot (x-2)(x+4)$

**Applications:**

Example 4: The rational expression  $y = \frac{250x}{100-x}$  models the cost, in millions of dollars, to remove  $x$  percent of the pollutants that are discharged into a river.

- a. How much does it cost to remove 50% of the pollutants?  $\rightarrow x=50$
- b. How much does it cost to remove 100% of the pollutants?  $\rightarrow x=100$

(a)  $y = \frac{250(50)}{100-(50)} \rightarrow y = \frac{250}{1}$   
 $y = \frac{250 \cdot (50)}{1} \rightarrow y = 250$

(b)  $y = \frac{250 \cdot (100)}{100-(100)}$   
 $y = \frac{250 \cdot (100)}{0} \leftarrow \text{undefined}$

It will cost  $\$250$  million to remove 50% of the pollutants.

100 is an excluded value.

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer. We cannot remove 100% of the pollutants.

## Answers Section 7.1

Example 1:

- Excluded value (value for which the expression is undefined) is 3.
- Excluded value is  $\frac{5}{2}$
- Excluded values are  $-2$  and  $3$ .
- Excluded values are  $-3$  and  $3$ .

Example 2:

- $\frac{x+7}{5x}$
- $\frac{5}{3}$
- $x^2$

Example 3:

- $-1$
- $-1$
- $-\frac{(2x+5)}{3}$  or  $-\frac{(2x+5)}{3}$
- $\frac{-(x-3)}{4+x}$  or  $-\frac{(x-3)}{4+x}$

Example 4:

- \$250 million
- The expression is not defined at 100%, so theoretically it is not possible to remove 100% of the pollutants.

## Section 7.2 Multiplying and Dividing Rational Expressions

**Multiplying Rational Expressions:** If  $P$ ,  $Q$ ,  $R$  and  $S$  are polynomials, where  $Q$  is not zero and  $S$  is not zero, then

$$\frac{P}{Q} \cdot \frac{R}{S} = \frac{PR}{QS}$$

Steps for multiplying rational expressions:

1. Factor all numerators and denominators completely.
2. Divide numerators and denominators by common factors.
3. Multiply the remaining factors in the numerators and multiply the remaining factors in the denominators.

NOTE: You do not need a common denominator to multiply rational expressions,

Example 1: Multiply the given expressions, and simplify your result.

a.  $\frac{x-2}{x+3} \cdot \frac{2x+6}{5x-10} = \frac{(x-2) \cdot 2 \cdot (x+3)}{(x+3) \cdot 5 \cdot (x-2)} = \frac{2}{5}$

b.  $\frac{x-3}{x+7} \cdot \frac{3x+21}{3x-9} = \frac{(x-3) \cdot 3 \cdot (x+7)}{(x+7) \cdot 3 \cdot (x-3)} = 1$

c.  $\frac{2y}{3y-y^2} \cdot \frac{2y^2-9y+9}{8y-12} = \frac{\cancel{2} \cdot \cancel{y} \cdot (2y-3) \cdot (y-3)}{-y \cdot (y-3) \cdot \cancel{2} \cdot 2 \cdot (2y-3)} = -\frac{1}{2}$

SIDEWORK

$$2x+6 = 2 \cdot x + 2 \cdot 3 = 2(x+3)$$

$$5x-10 = 5 \cdot x - 5 \cdot 2 = 5(x-2)$$

$$3x+21 = 3 \cdot x + 3 \cdot 7 = 3 \cdot (x+7)$$

$$3x-9 = 3 \cdot x - 3 \cdot 3 = 3(x-3)$$

$$2y^2-9y+9 = (2y-3)(y-3)$$

$$3y-y^2 = -y^2+3y = -y \cdot y + 3 \cdot y = -y \cdot (y-3)$$

$$8y-12 = 4 \cdot 2y - 4 \cdot 3 = 4 \cdot (2y-3) = 2 \cdot 2 \cdot (2y-3)$$

**Dividing Rational Expressions:** If P, Q, R and S are polynomials, where Q is not zero, R is not zero and S is not zero, then

$$\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \cdot \frac{S}{R} = \frac{PS}{QR}$$

To find the quotient of two rational expressions, invert the divisor and multiply.

Example 2: Divide the given expressions. Simplify your result.

a.  $\frac{x+5}{7} \div \frac{4x+20}{7} = \frac{x+5}{7} * \frac{7}{4x+20} = ?$   
 $= \frac{7 \cdot (x+5)}{7 \cdot 4 \cdot (x+5)} = \frac{1}{4}$

b.  $\frac{4}{x-6} \div \frac{40}{7x-42} = \frac{4}{x-6} \cdot \frac{7x-42}{40}$   
 $= \frac{2 \cdot 2 \cdot 7 \cdot (x-6)}{(x-6) \cdot 2 \cdot 2 \cdot 2 \cdot 5} = \frac{7}{10}$

c.  $\frac{x^2+x}{x^2-4} \div \frac{x^2-1}{x^2+5x+6} = \frac{x^2+x}{x^2-4} \cdot \frac{x^2+5x+6}{x^2-1}$   
 $= \frac{x \cdot (x+1) \cdot (x+2)(x+3)}{(x-2)(x+2)(x-1)(x+1)} = \frac{x(x+3)}{(x-2)(x-1)}$

d.  $\frac{3y+12}{y^2+3y} \div \frac{y^2+y-12}{9y-y^3} = \frac{3y+12}{y^2+3y} \cdot \frac{9y-y^3}{y^2+y-12}$   
 $= \frac{3 \cdot (y+4) \cdot (-y)(y+3) \cdot (y-3)}{y \cdot (y+3) \cdot (y-3)(y+4)} = \frac{3 \cdot (-1)}{1} = -3$

SDWK

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 $4x+20 = 4 \cdot x + 4 \cdot 5 = 4(x+5)$ 


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 $7x-42 = 7 \cdot x - 7 \cdot 6 = 7(x-6)$ 


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 $40 = 2 \cdot 2 \cdot 2 \cdot 5$ 

```

  40
  4  10
  2  2  5
  
```

---

 $x^2+x = x \cdot x + x \cdot 1 = x(x+1)$ 


---

 $x^2+5x+6 = (x+2)(x+3)$ 


---

 $x^2-4 = (x+2)(x-2)$ 


---

 $x^2-1 = (x-1)(x+1)$ 


---

 $3y+12 = 3 \cdot y + 3 \cdot 4 = 3 \cdot (y+4)$ 


---

 $9y-y^3 = -y^3+9y = -y(y^2-9) = -y(y+3)(y-3)$ 


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 $y^2+3y = y \cdot y + 3y = y(y+3)$ 


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 $y^2+y-12 = (y+4)(y-3)$ 

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12  
1,12  
2,6  
3,4

**Answers Section 7.2**

Example 1:

a.  $\frac{2}{5}$

b. 1

c.  $-\frac{1}{2}$

Example 2:

a.  $\frac{1}{4}$

b.  $\frac{7}{10}$

c.  $\frac{x(x+3)}{(x-2)(x-1)}$

d. -3

## Section 7.3 Adding and Subtracting Rational Expressions with the Same Denominators

### Adding Rational Expressions with Common Denominators:

If  $\frac{P}{R}$  and  $\frac{Q}{R}$  are rational expressions, then

$$\frac{P}{R} + \frac{Q}{R} = \frac{P+Q}{R}$$

To add rational expressions with the same denominators, add numerators and place the sum over the common denominator. If possible, simplify the final result. Recall that to simplify rational expressions, you should factor both numerator and denominator completely and then divide out common factors.

Example 1: Add. Express your result in simplest form.

a.  $\frac{x}{x-4} + \frac{9x+7}{x-4} = \frac{x+9x+7}{x-4} = \frac{10x+7}{x-4}$

b.  $\frac{4x+1}{6x+5} + \frac{8x+9}{6x+5} = \frac{4x+1+8x+9}{6x+5} = \frac{12x+10}{6x+5}$   
 $= \frac{2 \cdot (6x+5)}{1 \cdot (6x+5)} = \frac{2}{1} = 2$

SDUK

$$12x+10 = 2 \cdot 6x + 2 \cdot 5 \\ = 2 \cdot (6x+5)$$

### Subtracting Rational Expressions with Common Denominators:

If  $\frac{P}{R}$  and  $\frac{Q}{R}$  are rational expressions, then

$$\frac{P}{R} - \frac{Q}{R} = \frac{P-Q}{R}$$

To subtract rational expressions with the same denominators, subtract numerators and place the difference over the common denominator. If possible, simplify the final result. Recall that to simplify rational expressions, you should factor both numerator and denominator completely and then divide out common factors.



SDWK

$$4x-2 = 2 \cdot 2x - 2 \cdot 1 \\ = 2(2x-1)$$

Example 2: Subtract. Express your result in simplest form.

$$a. \frac{2x}{4x-2} - \frac{1}{4x-2} = \frac{2x-1}{4x-2} = \frac{1 \cdot (2x-1)}{2 \cdot (2x-1)} = \frac{1}{2}$$

$$b. \frac{x^3-3}{2x^4} - \frac{7x^3-3}{2x^4} = \frac{x^3-3-(7x^3-3)}{2x^4} \\ = \frac{x^3-3-7x^3+3}{2x^4} = \frac{-6x^3}{2x^4} = \frac{-2 \cdot 3 \cdot x \cdot x \cdot x}{2 \cdot x \cdot x \cdot x \cdot x} = \frac{-3}{x}$$

### Adding and Subtracting Rational Expressions with Opposite

**Denominators:** When one denominator is the additive inverse of the other (identical expressions except for opposite signs on each term),

first multiply either rational expression by  $\frac{-1}{-1}$  to obtain a common denominator and then add or subtract as indicated.

Example 3: Add or subtract as indicated. Express your results in simplest form.

$$a. \frac{4}{x-3} + \frac{2}{3-x} = \frac{4}{x-3} + \frac{-1(2)}{-1(3-x)} = \frac{4}{x-3} + \frac{-2}{x-3} = ? \\ = \frac{4-2}{x-3} = \frac{2}{x-3}$$

$$b. \frac{6x+5}{x-2} + \frac{4x}{2-x} = \frac{6x+5}{x-2} + \frac{(-1) \cdot (4x)}{(-1)(2-x)} \\ = \frac{6x+5}{x-2} + \frac{-4x}{x-2} = \frac{6x+5+(-4x)}{x-2} = \frac{2x+5}{x-2}$$

$$c. \frac{3-x}{x-7} - \frac{2x-5}{7-x} = \frac{3-x}{x-7} - \frac{(-1) \cdot (2x-5)}{(-1)(7-x)} \\ = \frac{3-x}{x-7} + \frac{2x-5}{x-7} = \frac{3-x+2x-5}{x-7} = \frac{x-2}{x-7}$$

SDWK

$$-1 \cdot (3-x) = -3+x \\ = x+(-3) \\ = x-3$$

$$-1 \cdot (2-x) = -2+x \\ = x+(-2) \\ = x-2$$

$$-1 \cdot (7-x) = -7+x \\ = x-7$$

**Answers Section 7.3**

Example 1:

a.  $\frac{10x+7}{x-4}$

b. 2

Example 2:

a.  $\frac{1}{2}$

b.  $\frac{-3}{x}$  or  $-\frac{3}{x}$

Example 3:

a.  $\frac{2}{x-3}$

b.  $\frac{2x+5}{x-2}$

c.  $\frac{x-2}{x-7}$

## Section 7.4 Adding and Subtracting Rational Expressions with Different Denominators

In order to add or subtract rational expressions that have different denominators, you must first find a least common denominator and then rewrite each rational expression as an equivalent rational expression that has the common denominator.

### Finding the Least Common Denominator

1. Factor each denominator completely.
2. List the factors of the first denominator.
3. Add to the list in step 2 any factors of the second denominator that do not appear in the list. (NOTE: If a factor appears once in the first denominator and more than once in the second denominator, then you must add to the list the "extra" factors from the second denominator.)
4. Form the product of each different factor from the list in step 3. This product is the least common denominator.

Example 1: Find the least common denominator for each of the following pairs of rational expressions.

a.  $\frac{7}{15x^2}, \frac{13}{24x}$

Factors of 1st denominator: 3, 5, x, x

Factors of 2nd denominator: 2, 2, 2, 3, x

Add to the first list any "missing" factor from the 2nd list: 3, 5, x, x, 2, 2, 2

$$\text{LCD} = 3 * 5 * x * x * 2 * 2 * 2 = 120x^2$$

b.  $\frac{2}{x-5}, \frac{3}{x^2-25}$

c.  $\frac{3}{x^2-25}, \frac{x}{x^2-10x+25}$

$$\begin{aligned} \text{LCD} &= (x-5) \cdot (x+5) \\ x-5 &= 1 \cdot (x-5) \\ x^2-25 &= (x+5)(x-5) \end{aligned}$$

$$\begin{aligned} \text{LCD} &= (x-5)(x+5)(x-5) = (x-5)^2(x+5) \\ x^2-25 &= (x-5)(x+5) \\ x^2-10x+25 &= (x-5)(x-5) \end{aligned}$$

**Adding and Subtracting Rational Expressions That Have Different Denominators:** To add or subtract rational expressions that have different denominators:

1. Find the LCD of the rational expressions.
2. Rewrite each rational expression as an equivalent expression whose denominator is the LCD. To do so, multiply the numerator and denominator of each rational expression by any factor(s) needed to convert the denominator into the LCD.
3. Add or subtract the numerators, placing the resulting expression over the LCD.
4. If necessary, simplify the resulting expression.

SDWK  
 LCD = 2 · 3 · x · 2 · 2  
 6x = 2 · 3 · x  
 8x = 2 · 2 · 2 · x  
 LCD = 24x

Example 2: Add or subtract, as indicated. Express your result in simplest form.

LCD = x · 1 = x  
 x = x  
 1 = 1

$$a. \frac{5}{6x} + \frac{7}{8x} = \frac{5}{6x} \cdot \frac{4}{4} + \frac{7}{8x} \cdot \frac{3}{3} = \frac{20}{24x} + \frac{21}{24x} = \frac{41}{24x}$$

$$b. \frac{5}{x} + 3 = \frac{5}{x} + \frac{3 \cdot x}{1 \cdot x} = \frac{5}{x} + \frac{3x}{x} = \frac{5+3x}{x}$$

$$c. \frac{4x}{x^2-25} + \frac{x}{x+5} = \frac{4x}{(x-5)(x+5)} + \frac{x}{(x+5)} \cdot \frac{(x-5)}{(x-5)} = \frac{4x}{(x-5)(x+5)} + \frac{x^2-5x}{(x-5)(x+5)} = \frac{4x+x^2-5x}{(x-5)(x+5)} = \frac{x^2-x}{(x-5)(x+5)} = \frac{x^2-x}{x^2-25}$$

$$d. \frac{x}{x^2-2x-24} - \frac{x}{x^2-7x+6} = \frac{x}{(x-6)(x+4)} \cdot \frac{(x-1)}{(x-1)} + \frac{-x}{(x-6)(x-1)} \cdot \frac{(x+4)}{(x+4)} = \frac{x^2-x}{(x-6)(x+4)(x-1)} + \frac{-x^2-4x}{(x-6)(x+4)(x-1)} = \frac{x^2-x-x^2-4x}{(x-6)(x+4)(x-1)} = \frac{-5x}{(x-6)(x+4)(x-1)}$$

LCD = (x-5)(x+5)  
 x^2-25 = (x-5)(x+5)  
 x+5 = 1 · (x+5)

LCD = (x-6)(x+4)(x-1)  
 x^2-2x-24 = (x-6)(x+4)

24
1, 24
2, 12
3, 8
4, 6

x^2-7x+6 = (x-6)(x-1)

6
1, 6
2, 3

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer.

### Adding and Subtracting Rational Expressions When

**Denominators Contain Opposite Factors:** When one denominator contains the opposite factor of the other, first multiply either rational

expression by  $\frac{-1}{-1}$ . Then apply the procedure for adding or

subtracting rational expressions that have different denominators to the rewritten problem.

**Example 3:** Add or subtract, as indicated. Express your result in simplest form.

$$\begin{aligned} \text{a. } \frac{x+7}{9x-12} + \frac{x}{16-9x^2} &= \frac{x+7}{9x-12} + \frac{(-1)}{(-1)} \cdot \frac{x}{(16-9x^2)} \\ &= \frac{x+7}{9x-12} + \frac{-x}{9x^2-16} = \frac{(x+7)}{3(3x-4)} \cdot \frac{(3x+4)}{(3x+4)} + \frac{-x}{(3x-4)(3x+4)} \cdot \frac{3}{3} \\ &= \frac{3x^2+25x+28 + (-3x)}{3 \cdot (3x-4)(3x+4)} = \frac{3x^2+22x+28}{3(3x-4)(3x+4)} \end{aligned}$$

$$\begin{aligned} \text{b. } \frac{y}{y^2-1} + \frac{5y}{y-y^2} &= \frac{y}{y^2-1} + \frac{(-1)}{(-1)} \cdot \frac{5y}{(y-y^2)} \\ &= \frac{y}{y^2-1} + \frac{-5y}{y^2-y} = \frac{y}{(y+1)(y-1)} \cdot \frac{y}{y} + \frac{-5y}{y \cdot (y-1)} \cdot \frac{(y+1)}{(y+1)} \\ &= \frac{y^2-5y^2-5y}{y(y+1)(y-1)} = \frac{-4y^2-5y}{y(y+1)(y-1)} = \frac{-y \cdot (4y+5)}{y(y+1)(y-1)} \\ &= \frac{-(4y+5)}{(y+1)(y-1)} \end{aligned}$$

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$$\begin{aligned} -1 \cdot (16-9x^2) \\ &= -16 + 9x^2 \\ &= 9x^2 - 16 \end{aligned}$$

$$\begin{aligned} 9x-12 &= 3 \cdot 3x - 3 \cdot 4 \\ &= 3 \cdot (3x-4) \end{aligned}$$

$$9x^2-16 = (3x-4)(3x+4)$$

$$\text{LCD} = 3(3x-4)(3x+4)$$

$$\begin{aligned} (x+7)(3x+4) \\ &= 3x^2 + 4x + 21x + 28 \\ &= 3x^2 + 25x + 28 \end{aligned}$$

$$3x^2 + 22x + 28$$

$$= (3x \quad x \quad )$$

$$\text{Prime} \quad 28$$

$$\text{Doesn't} \quad 1, 28$$

$$\text{factor!} \quad 2, 14$$

$$4, 7$$

$$-1 \cdot (y-y^2) = -y + y^2$$

$$= y^2 - y$$

$$y^2-1 = (y-1)(y+1)$$

$$y^2-y = y \cdot (y-1)$$

$$\text{LCD} = y(y-1)(y+1)$$

$$-5y \cdot (y+1) = -5y^2 - 5y$$

$$-4y^2 - 5y$$

$$= -y \cdot 4y - y \cdot 5$$

$$= -y \cdot (4y + 5)$$

**Answers Section 7.4**

Example 1:

- a. LCD is  $120x^2$
- b. LCD is  $(x - 5)(x + 5)$
- c. LCD is  $(x - 5)(x + 5)(x - 5)$

Example 2:

- a.  $\frac{41}{24x}$
- b.  $\frac{5+3x}{x}$  or  $\frac{3x+5}{x}$
- c.  $\frac{x^2 - x}{x^2 - 25}$
- d.  $\frac{-5x}{(x-6)(x+4)(x-1)}$

Example 3:

- a.  $\frac{3x^2 + 22x + 28}{3(3x-4)(4+3x)}$
- b.  $\frac{-(4y+5)}{(y+1)(y-1)}$

## 7.5 Complex Rational Expressions

Complex rational expressions have numerators or denominators containing one or more rational expressions.

### Simplifying a Complex Rational Expression by Dividing:

Follow these steps to simplify a complex rational expression by dividing:

1. If necessary, add or subtract to get a single rational expression in the numerator.
2. If necessary, add or subtract to get a single rational expression in the denominator.
3. Perform the division indicated by the main fraction bar: invert the denominator of the complex rational expression and multiply.
4. If possible, simplify.

Example 1: Simplify each complex rational expression by using the division method.

$$\begin{aligned} \text{a. } 5 - \frac{2}{x} &= \left(\frac{5}{1} - \frac{2}{x}\right) \div \left(\frac{3}{1} + \frac{1}{x}\right) && \rightarrow = \left(\frac{5x-2}{x}\right) \div \left(\frac{3x+1}{x}\right) \\ & && = \left(\frac{5x-2}{x}\right) \cdot \left(\frac{x}{3x+1}\right) \\ 3 + \frac{1}{x} &= \left(\frac{5}{1} \cdot \frac{x}{x} - \frac{2}{x}\right) \div \left(\frac{3}{1} \cdot \frac{x}{x} + \frac{1}{x}\right) && = \frac{5x-2}{3x+1} \end{aligned}$$

$$\begin{aligned} \text{b. } 4 - \frac{7}{y} &= \left(\frac{4}{1} - \frac{7}{y}\right) \div \left(\frac{3}{1} - \frac{2}{y}\right) && \rightarrow = \left(\frac{4y-7}{y}\right) \div \left(\frac{3y-2}{y}\right) \\ & && = \left(\frac{4y-7}{y}\right) \cdot \left(\frac{y}{3y-2}\right) \\ 3 - \frac{2}{y} &= \left(\frac{4}{1} \cdot \frac{y}{y} - \frac{7}{y}\right) \div \left(\frac{3}{1} \cdot \frac{y}{y} - \frac{2}{y}\right) && = \frac{4y-7}{3y-2} \end{aligned}$$

$$\begin{aligned} \text{c. } \frac{8}{x^2} - \frac{2}{x} &= \left(\frac{8}{x^2} - \frac{2}{x}\right) \div \left(\frac{10}{x} - \frac{6}{x^2}\right) && \rightarrow = \left(\frac{8-2x}{x^2}\right) \div \left(\frac{10x-6}{x^2}\right) \\ & && = \left(\frac{8-2x}{x^2}\right) \cdot \left(\frac{x^2}{10x-6}\right) \\ \frac{10}{x} - \frac{6}{x^2} &= \left(\frac{8}{x^2} - \frac{2}{x} \cdot \frac{x}{x}\right) \div \left(\frac{10}{x} \cdot \frac{x}{x} - \frac{6}{x^2}\right) && = \frac{-2 \cdot (x-4)}{2 \cdot (5x-3)} = -\frac{x-4}{5x-3} \end{aligned}$$

$$\begin{aligned} \text{d. } \frac{3}{x+2} - \frac{3}{x-2} &= \left(\frac{3}{x+2} - \frac{3}{x-2}\right) \div \left(\frac{5}{x^2-4}\right) && \rightarrow = \left[\frac{3x-6-3x-6}{(x+2)(x-2)}\right] \div \left(\frac{5}{x^2-4}\right) \\ & && = \frac{-12}{(x+2)(x-2)} \cdot \frac{x^2-4}{5} = \frac{-12(x-2)(x+2)}{5(x+2)(x-2)} \\ & && = \frac{-12}{5} \end{aligned}$$

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer.

SDMK

LCD = x

LCD = y

LCD = x^2

$$\begin{aligned} 8-2x &= -2x+8 \\ &= -2x+(2x+4) \\ &= -2 \cdot (x-4) \end{aligned}$$

$$\begin{aligned} 10x-6 &= 2 \cdot 5x-2 \cdot 3 \\ &= 2 \cdot (5x-3) \end{aligned}$$

LCD = (x+2)(x-2)

x^2-4 = (x-2)(x+2)

= -12/5

## Simplifying a Complex Rational Expression by Multiplying by the LCD:

Follow these steps to simplify a complex rational expression by multiplying by the LCD:

1. Find the LCD of all rational expressions within the complex rational expression.
2. Multiply both the numerator and the denominator of the complex rational expression by this LCD.
3. Use the distributive property and multiply each term in the numerator and each term in the denominator by the LCD. Simplify. No fractional expressions should remain (that is, the numerator should be a polynomial and the denominator should be a polynomial.) This process is called clearing fractions.
4. If possible, factor and simplify.

Example 2: Simplify by multiplying by the LCD.

LCD = x

$$a. \frac{5 - \frac{2}{x}}{3 + \frac{1}{x}} = \left( \frac{\frac{5}{1} - \frac{2}{x}}{\frac{3}{1} + \frac{1}{x}} \right) \cdot \left( \frac{x}{1} \right) = \frac{\frac{5x}{1} - \frac{2x}{x}}{\frac{3x}{1} + \frac{x}{x}} = \frac{5x-2}{3x+1}$$

LCD = y

$$b. \frac{4 - \frac{7}{y}}{3 - \frac{2}{y}} = \left( \frac{\frac{4}{1} - \frac{7}{y}}{\frac{3}{1} - \frac{2}{y}} \right) \cdot \left( \frac{y}{1} \right) = \frac{\frac{4y}{1} - \frac{7y}{y}}{\frac{3y}{1} - \frac{2y}{y}} = \frac{4y-7}{3y-2}$$

LCD = x<sup>2</sup>

$$c. \frac{\frac{8}{x^2} - \frac{2}{x}}{\frac{10}{x} - \frac{6}{x^2}} = \left( \frac{\frac{8}{x^2} - \frac{2}{x}}{\frac{10}{x} - \frac{6}{x^2}} \right) \cdot \left( \frac{x^2}{1} \right) = \frac{\frac{8x^2}{x^2} - \frac{2x^2}{x}}{\frac{10x^2}{x} - \frac{6x^2}{x^2}} = \frac{8-2x}{10x-6} = \frac{-2(x-4)}{2(5x-3)} = \frac{-(x-4)}{5x-3}$$

LCD = (x+2)(x-2)

$$d. \frac{\frac{3}{x+2} - \frac{3}{x-2}}{\frac{5}{x^2-4}} = \left[ \frac{\frac{3}{x+2} - \frac{3}{x-2}}{5} \right] \cdot \left[ \frac{(x+2)(x-2)}{1} \right] = \frac{3(x-2) - 3(x+2)}{5} = \frac{3x-6-3x-6}{5} = \frac{-12}{5}$$

$$\begin{array}{l} \text{SDWK} \\ \hline 8-2x = -2x+8 \\ \quad = -2(x-4) \\ \hline 10x-6 = 2(5x-3) \\ \hline x^2-4 = (x+2)(x-2) \end{array}$$



**Answers Section 7.5**

Example 1:

a.  $\frac{5x-2}{3x+1}$

b.  $\frac{4y-7}{3y-2}$

c.  $\frac{4-x}{5x-3}$  or  $-\frac{x-4}{5x-3}$

d.  $\frac{-12}{5}$  or  $-\frac{12}{5}$

Example 2: The answers are the same as for example 1.

## Section 7.6 Solving Rational Equations

### Solving Rational Equations

Follow these steps to solve a rational equation:

1. List restrictions on the variable. Avoid any values of the variable that make a denominator zero.
2. Clear the equation of fractions by multiplying both sides by the LCD of all rational expressions in the equation.
3. Solve the resulting equation.
4. Reject any proposed solution that is in the list of restrictions on the variable. Check other proposed solutions in the original equation.

work

$$x^2 - 16 = (x-4)(x+4)$$

$$y^2 + 7y - 8 = (y-8)(y+1)$$

Example 1: Solve each of the following rational equations.

a.  $x + \frac{3}{x} = \frac{19}{x}$

$$x \left[ x + \frac{3}{x} \right] = x \cdot \frac{19}{x}$$

$$x^2 + 3 = 19$$

$$-19 + x^2 + 3 = -19 + 19$$

$$x^2 - 16 = 0$$

$$(x-4)(x+4) = 0$$

Either

$$x-4 = 0, \text{ or } x+4 = 0$$

$$x = 4 \quad x = -4$$

check:  $\frac{4}{4} + \frac{3}{4} = \frac{19}{4}$

$$\frac{4}{1} \cdot \left[ \frac{4}{1} + \frac{3}{4} \right] = \frac{4}{1} \cdot \frac{19}{4}$$

$$\frac{16}{1} + 3 = 19$$

$$16 + 3 = 19$$

$$19 = 19$$

TRUE!

$$\frac{4}{1} + \frac{3}{-4} = \frac{19}{-4}$$

$$\frac{-4}{1} \cdot \left[ -4 - \frac{3}{4} \right] = \frac{-4}{1} \cdot \left( \frac{19}{-4} \right)$$

$$16 + 3 = 19$$

$$19 = 19$$

TRUE!

AUS:

The solution set is  $\{4, -4\}$

LCD = 2y

b.  $\frac{4}{y} - \frac{y}{2} = \frac{7}{2}$

$$\frac{2y}{1} \cdot \left[ \frac{4}{y} - \frac{y}{2} \right] = \frac{2y}{1} \cdot \frac{7}{2}$$

$$8 - y^2 = 7y$$

$$(y^2 - 8) + 8 - y^2 = 7y + (y^2 - 8)$$

$$0 = y^2 + 7y - 8$$

$$0 = (y+8)(y-1)$$

Either

$$y+8 = 0, \text{ or } y-1 = 0$$

$$y = -8 \quad y = 1$$

check:

$$\frac{4}{-8} - \frac{-8}{2} = \frac{7}{2}$$

$$-\frac{1}{2} + \frac{4}{1} = \frac{7}{2}$$

$$-\frac{1}{2} + \frac{8}{2} = \frac{7}{2}$$

$$\frac{7}{2} = \frac{7}{2}$$

TRUE!

$$\frac{4}{1} - \frac{1}{2} = \frac{7}{2}$$

$$\frac{4}{1} \cdot \frac{2}{2} - \frac{1}{2} = \frac{7}{2}$$

$$\frac{8}{2} - \frac{1}{2} = \frac{7}{2}$$

$$\frac{7}{2} = \frac{7}{2}$$

TRUE!

AUS:

The solution set is  $\{-8, 1\}$

19/29

SDWK
$2y-2=2 \cdot (y-1)$

LC) = 2 \cdot (y-1)

c.  $\frac{3}{2y-2} + \frac{1}{2} = \frac{2}{y-1}$

$\frac{2(y-1)}{1} \cdot \left[ \frac{3}{2 \cdot (y-1)} + \frac{1}{2} \right] = \frac{2 \cdot (y-1)}{1} \cdot \left( \frac{2}{y-1} \right)$

$3 + 1(y-1) = 4$   
 $y + 2 = 4$   
 $-2 + y + 2 = -2 + 4$   
 $y = 2$

check!

$\frac{3}{2(2)-2} + \frac{1}{2} = \frac{2}{(2)-1}$   
 $\frac{3}{4-2} + \frac{1}{2} = \frac{2}{1}$   
 $\frac{3}{2} + \frac{1}{2} = 2$   
 $\frac{4}{2} = 2$   
 $2 = 2$   
 TRUE!

ANS!  
The solution set is {2}

SDWK
$x^2+x-6$
$= (x+3)(x-2)$

LCD = (x-2)(x+3)

d.  $\frac{x-3}{x-2} + \frac{x+1}{x+3} = \frac{2x^2-15}{x^2+x-6}$

$\frac{(x-2)(x+3)}{1} \cdot \left[ \frac{x-3}{x-2} + \frac{x+1}{x+3} \right] = \frac{(x-2)(x+3)}{1} \cdot \left[ \frac{2x^2-15}{(x-2)(x+3)} \right]$

$(x-3)(x+3) + (x+1)(x-2) = 2x^2-15$   
 $x^2+3x-3x-9 + x^2-2x+x-2 = 2x^2-15$   
 $2x^2-x-11 = 2x^2-15$   
 $-2x^2+15 + 2x^2-x-11 = -2x^2+15 + 2x^2-15$   
 $-x+4 = 0$   
 $-x+4+x = 0+x$   
 $4=x$

check!

$\frac{(4)-3}{(4)-2} + \frac{(4)+1}{(4)+3} = \frac{2(4)^2-15}{(4)^2+(4)-6}$   
 $\frac{1}{2} + \frac{5}{7} = \frac{2 \cdot 16 - 15}{16 + 4 - 6}$   
 $\frac{1}{2} \cdot \frac{7}{7} + \frac{5}{7} \cdot \frac{2}{2} = \frac{32-15}{20-6}$   
 $\frac{7}{14} + \frac{10}{14} = \frac{17}{14}$   
 $\frac{17}{14} = \frac{17}{14}$   
 TRUE!

ANS!  
The solution set is {4}

### Applications of Rational Equations:

To Solve Applied Problems Using Rational Equations:

1. Identify the quantity represented by each variable in the rational equation.
2. Plug the known quantities into the equation for the appropriate variables.
3. Solve for the unknown variable.

Example 2: The rational expression  $y = \frac{250x}{100 - x}$  models the cost, in millions of dollars, to remove  $x$  percent of the pollutants that are discharged into a river.

- a. How much does it cost to remove 50% of the pollutants?
- b. If the government commits \$375 million for this project, what percentage of the pollutants can be removed?

(a)  $x = 50$

Find  $y$ :

$$y = \frac{250 \cdot (50)}{100 - (50)}$$

$$y = \frac{250 \cdot (50)}{50}$$

$$y = 250$$

**ANS:** It should cost approximately 250 million dollars to remove 50% of the pollutants.

(b)  $y = 375$

Find  $x$ :

$$(375) = \frac{250x}{100 - x}$$

$$375 \cdot \frac{(100 - x)}{1} = \left( \frac{250x}{100 - x} \right) \cdot \left( \frac{100 - x}{1} \right)$$

$$37,500 - 375x = 250x$$

$$37,500 - 375x + 375x = 250x + 375x$$

$$37,500 = 625x$$

$$\frac{37,500}{625} = \frac{625x}{625}$$

$$60 = x$$

check:

$$375 = \frac{250(60)}{100 - (60)}$$

$$375 = \frac{15,000}{40}$$

$$375 = 375$$

TRUE!

**ANS:** Approximately 60% of the pollutants can be removed for \$375 million.

**Answers Section 7.6**

Example 1:

- a.  $\{4, -4\}$
- b.  $\{-8, 1\}$
- c.  $\{2\}$
- d.  $\{4\}$

Example 2:

- a. \$250 million
- b. 60% will be removed

## Section 7.7 Applications Using Rational Equations and Proportions

### Solving Application Problems

Follow these steps to properly show your work when solving an application problem.

Step 1: Let  $x$  represent one of the unknown quantities.

Step 2: Represent any other unknown quantities in terms of  $x$ .

Step 3: Write an equation that describes the conditions.

Step 4: Solve the equation and answer the question.

Step 5: Check the proposed solution in the original wording of the problem.

### Problems Involving Motion

To solve problems involving motion, the formula that relates time traveled to the distance traveled and the rate of travel is needed. The formula is:

$$\text{Time traveled} = \frac{\text{Distance traveled}}{\text{Rate of travel}}$$

Use a chart to organize the information in the problem.

Example 1: You can travel 40 miles on a motorcycle in the same time that it takes to travel 15 miles on a bicycle. If your motorcycle's rate is 20 miles per hour faster than your bicycle's, find the average rate for each.

*This will be part of your grade.*

Let  $x$  be the rate of the bicycle.

	Distance	Rate	Time = $\frac{\text{Distance}}{\text{Rate}}$
Motorcycle	40	$x + 20$	$\frac{40}{x + 20}$
Bicycle	15	$x$	$\frac{15}{x}$

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer.

Now, write your equation by setting the times equal to each other:

$$\frac{40}{x+20} = \frac{15}{x}$$

Solve this equation, and then express your result in English words. LCD =  $x \cdot (x+20)$

$$\frac{x \cdot (x+20)}{1} \cdot \left[ \frac{40}{x+20} \right] = \frac{x \cdot (x+20)}{1} \cdot \left[ \frac{15}{x} \right]$$

$$x \cdot 40 = (x+20) \cdot 15$$

$$40x = 15x + 300$$

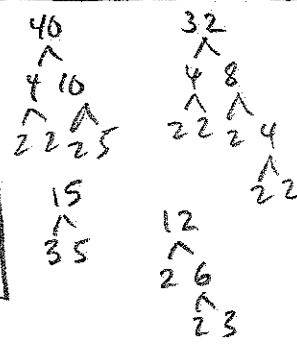
$$-15x + 40x = -15x + 15x + 300$$

$$25x = 300$$

$$\frac{1}{25} \cdot \frac{25x}{1} = \frac{1}{25} \cdot \frac{300}{1}$$

$$x = 12$$

sdwk



check!

$$\frac{40}{(12)+20} = \frac{15}{(12)}$$

$$\frac{40}{32} = \frac{15}{12}$$

$$\frac{2 \cdot 2 \cdot 2 \cdot 5}{2 \cdot 2 \cdot 2 \cdot 2} = \frac{3 \cdot 5}{2 \cdot 2 \cdot 3}$$

$$\frac{5}{4} = \frac{5}{4}$$

TRUE!

Motorcycle's Rate =  $(12)+20$   
= 32 miles per hour

bicycle's Rate =  $(12)$   
= 12 miles per hour

ANS: The average rate for the motorcycle is 32 mph, and the average rate for the bicycle is 12 mph.

Example 2: The water's current is 2 miles per hour. A canoe can travel 6 miles downstream, with the current, in the same amount of time it travels 2 miles upstream, against the current. What is the canoe's average rate in still water?

Write a statement identifying what  $x$  represents, set up a chart to organize the given information, then use the information in the problem and the chart to write an equation. Solve the equation and write your result in English words.

Let  $x$  = the canoe's average rate in still water

	Distance	Rate	Time = $\frac{D}{R}$
upstream trip	2 miles	$(x-2)$ mph	$\frac{2}{x-2}$
downstream trip	6 miles	$(x+2)$ mph	$\frac{6}{x+2}$

Solve:

$$\frac{2}{x-2} = \frac{6}{x+2}$$

$$\text{LCD} = (x-2)(x+2)$$

$$\frac{(x-2)(x+2)}{1} \cdot \left[ \frac{2}{x-2} \right] = \frac{(x-2)(x+2)}{1} \cdot \left[ \frac{6}{x+2} \right]$$

$$(x+2) \cdot 2 = (x-2) \cdot 6$$

$$2x + 4 = 6x - 12$$

$$-2x + 2x + 4 = -2x + 6x - 12$$

$$4 = 4x - 12$$

$$12 + 4 = 12 + 4x - 12$$

$$16 = 4x$$

$$\frac{1}{4} \cdot \frac{16}{1} = \frac{1}{4} \cdot \frac{4x}{1}$$

$$4 = x$$

check:

$$\frac{2}{(4)-2} = \frac{6}{(4)+2}$$

$$\frac{2}{2} = \frac{6}{6}$$

$$1 = 1$$

TRUE!

Ans: The canoe's average rate in still water is 4 miles per hour.



### Problems Involving Work

To solve problems involving work, the formula that relates the sum of the portions of the job done by each participant to the complete job is needed. The formula is:

$$\begin{array}{l} \text{Fractional part} \\ \text{of the job done by} \\ \text{the first person} \end{array} + \begin{array}{l} \text{Fractional part} \\ \text{of the job done by} \\ \text{second person} \end{array} = 1$$

Number of Jobs

Use charts to organize the information in the problem.

Example 3: A pool can be filled by one pipe in 3 hours and by a second pipe in 6 hours. How long will it take using both pipes to fill the pool?

Let x be the number of hours to fill the pool when both pipes are in use.

	<i>"Rate of Work"</i> Fractional part of pool filled in one hour	Time working together	<i>"Amount of Work Done"</i> Fractional part of pool filled in x hours
Pipe #1	$\frac{1}{3}$	x	$\frac{x}{3}$
Pipe #2	$\frac{1}{6}$	x	$\frac{x}{6}$

Now write an equation, solve the equation, and write your result in English words.

Solve;

$$\frac{x}{3} + \frac{x}{6} = 1$$

$$\frac{x}{3} + \frac{x}{6} = 1$$

$$, \text{LCD} = 6$$

$$\frac{6}{1} \cdot \left[ \frac{x}{3} + \frac{x}{6} \right] = \frac{6}{1} \cdot 1$$

$$\frac{6}{1} \cdot \frac{x}{3} + \frac{6}{1} \cdot \frac{x}{6} = 6$$

$$2x + x = 6$$

$$3x = 6$$

$$\frac{1}{3} \cdot \frac{3x}{1} = \frac{1}{3} \cdot \frac{6}{1}$$

$$x = 2$$

check:

$$\frac{(2)}{3} + \frac{(2)}{6} = 1$$

$$\frac{2}{3} + \frac{1}{3} = 1$$

$$\frac{3}{3} = 1$$

$$1 = 1$$

TRUE!

sdwks

$$\frac{6}{2 \cdot 3}$$

$$\frac{2}{6} = \frac{1 \cdot 2}{2 \cdot 3} = \frac{1}{3}$$

$$\frac{6}{3} = \frac{2 \cdot 3}{1 \cdot 2} = 2$$

ANS: Working together, it will take 2 hours for both pipes to fill the pool.

### Problems Involving Proportions

A proportion is a statement that equates two ratios.

#### Cross-Products Principle for Proportions

If  $\frac{a}{b} = \frac{c}{d}$ , then  $ad = bc$  ( $b \neq 0$  and  $d \neq 0$ )

The cross products  $ad$  and  $bc$  are equal.

#### Solving Applied Problems Using Proportions

1. Read the problem, identify the quantity that you are solving for, and let  $x$  stand for that quantity.
2. Set up a proportion using the given ratio on one side and the ratio containing the unknown quantity on the other side.
3. Ignoring units, apply the cross-products principle.
4. Solve for  $x$  and answer the question in English words.

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer.

Example 4: The maintenance bill for a shopping center containing 180,000 square feet is \$45,000. What is the bill for a store in the center that has 4800 square feet? (Assume the bill is shared proportionally by all of the stores.)

Let  $x$  be the bill for the 4800 square foot store.

$$\frac{\$45,000}{180,000} = \frac{x}{4800}$$

"Use units"  $\rightarrow$   $\frac{\$45,000 \text{ maintenance}}{180,000 \text{ sq. ft.}} = \frac{\$x \text{ maintenance}}{4,800 \text{ sq. ft.}}$

Solve the equation and write your result in English words.

$$(45,000) \cdot (4,800) = (x) \cdot (180,000)$$

$$216,000,000 = 180,000x$$

$$\frac{216,000,000}{180,000} = \frac{180,000x}{180,000}$$

$$1,200 = x$$

check:

$$\frac{45,000}{180,000} = \frac{(1,200)}{4,800}$$

$$\frac{45}{180} = \frac{12}{48}$$

$$45 \cdot 48 = 12 \cdot 180$$

$$2,160 = 2,160$$

TRUE!

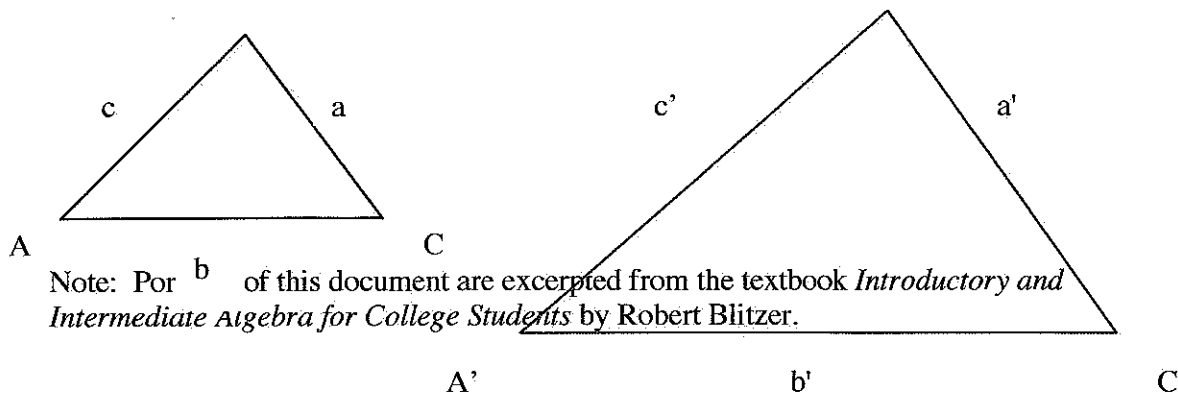
ANS: The maintenance bill for a 4,800 square foot store should be \$1,200.

### Problems Involving Similar Triangles

In similar triangles, the measures of corresponding angles are equal, and corresponding sides are proportional.

In the similar triangles below, the angles  $A$  and  $A'$  are equal, the angles  $B$  and  $B'$  are equal and the angles  $C$  and  $C'$  are equal. The corresponding sides are proportional:

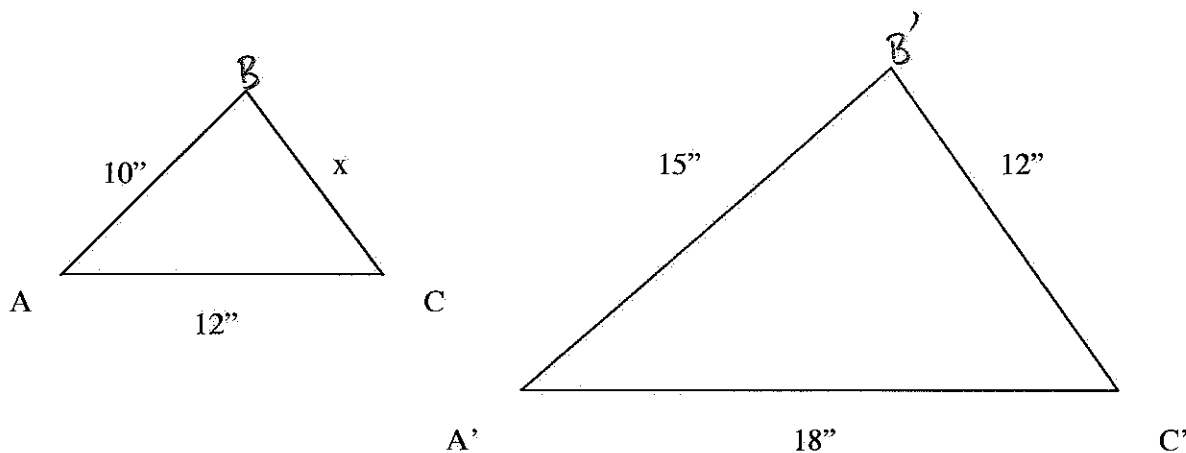
$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$



Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer.

Example 5: Use similar triangles and the fact that corresponding sides are proportional to find the length of the side marked with an "X".

Let  $x = \text{length of } BC$



Solve:

$$\frac{10}{15} = \frac{x}{12}$$

$$10 \cdot 12 = 15 \cdot x$$

$$120 = 15x$$

$$\frac{120}{15} = \frac{15x}{15}$$

$$8 = x$$

check:

$$\frac{10}{15} = \frac{8}{12}$$

$$10 \cdot 12 = 8 \cdot 15$$

$$120 = 120$$

TRUE!

ANS: The length of side BC is  $8''$ , or 8 inches.

**Answers Section 7.7**

Example 1: The rate of the motorcycle is 32 mph, and the rate of the bicycle is 12 mph.

Example 2: The rate of the canoe in still water is 4 mph.

Example 3: Together the pipes fill the pool in 2 hours.

Example 4: The maintenance bill for the 4800 square foot store is \$1200.

Example 5: The length of side "x" is 8".